

Perfect Actions for Scalar Theories *

W. Bietenholz, HLRZ c/o Forschungszentrum Jülich, 52425 Jülich, Germany

We construct an optimally local perfect lattice action for free scalars of arbitrary mass, and truncate its couplings to a unit hypercube. Spectral and thermodynamic properties of this “hypercube scalar” are drastically improved compared to the standard action. We also discuss new variants of perfect actions, using anisotropic or triangular lattices, or applying new types of RGTs. Finally we add a $\lambda\phi^4$ term and address perfect lattice perturbation theory. We report on a lattice action for the anharmonic oscillator, which is perfect to $O(\lambda)$.

Many examples have shown that the perfect action program to construct improved lattice actions works beautifully, if it can be properly implemented. However, a convincing application to QCD is still outstanding. Such attempts are a desperate struggle for locality and questions of parameterization and truncation are major issues. Hence it is motivated to study the properties of perfect action carefully in simple situations.

For free and perturbatively interacting fields, perfect action can be constructed conveniently by a technique that we call “blocking from the continuum” [1]. It corresponds to a blocking factor n RGT in the limit $n \rightarrow \infty$, so that the RGT does not need to be iterated. For a scalar we can relate the continuum resp. lattice field φ , ϕ as

$$\phi_x \sim \int \Pi(x-y)\varphi(y)dy, \quad (1)$$

where $x \in \mathbb{Z}^d$ and $\Pi(u) \doteq \prod_{\mu=1}^d \Theta(1/2 - |u_\mu|)$.

If we implement this relation by a Gaussian RGT term – with coefficient $1/\alpha$ – then we obtain in momentum space the perfect lattice action

$$S[\phi] = \frac{1}{(2\pi)^d} \int_B dk \frac{1}{2} \phi(-k) G^{-1}(k) \phi(k),$$

$$G(k) = \sum_{l \in \mathbb{Z}^d} \frac{\Pi^2(k + 2\pi l)}{(k + 2\pi l)^2 + m^2} + \alpha, \quad (2)$$

where $\Pi(k) = \prod_\mu \hat{k}_\mu/k_\mu$, $\hat{k}_\mu = 2 \sin(k_\mu/2)$ and $B =]-\pi, \pi]^d$. Remarkably, the spectrum of this action, $E^2(\vec{k}) = (\vec{k} + 2\pi\vec{l})^2 + m^2$, is exactly the continuum spectrum, which shows that the perfect action can reproduce the full Poincaré symmetry in observables, even though this symmetry is not manifest in the action.

*Talk presented at LAT97.

If we choose the RGT parameter $\alpha = \bar{\alpha}(m) \doteq (\sinh(m) - m)/m^3$, then the couplings in

$$S[\phi] = \frac{1}{2} \sum_{x,y} \phi_x \rho(x-y) \phi_y \quad (3)$$

are restricted to nearest neighbors in $d = 1$ [2], and they decay exponentially and very fast in $d > 1$.² The question arises, if this choice is really *optimal* for locality in $d = 4$, as we postulated before for fermions [1]. We measure the decay by $\rho(i, 0, 0, 0) \propto \exp\{-c(\alpha)i\}$, and Fig. 1 shows $c(\alpha)$ for various masses. Indeed, the peaks are just at $\bar{\alpha}(m)$.³

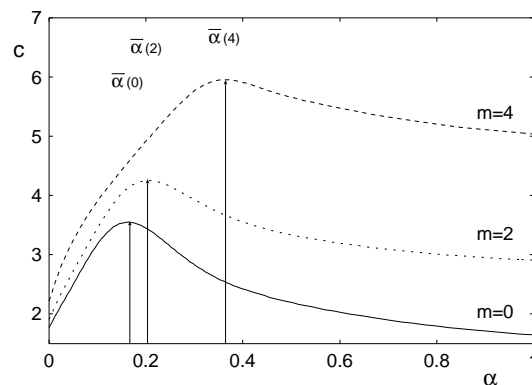


Figure 1. The locality parameter $c(\alpha)$.

As in the case of fermions, locality becomes even better for increasing mass, if we use $\bar{\alpha}(m)$, see Fig. 2.

To make this action applicable we need to truncate the couplings to a short range. We do so by

²For a finite blocking factor n and $m = 0$, $\bar{\alpha}$ is replaced by $\bar{\alpha}_n = (1 - 1/n^2)/6$, in agreement with [3].

³For heavy fermions, on the other hand, we noticed that locality can be improved slightly beyond the 1d formula.

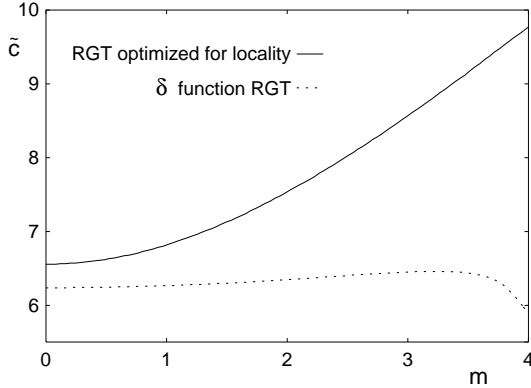


Figure 2. The locality measured by $\tilde{c}(m)$ in $\rho(i, i, i, i) \propto \exp(-\tilde{c}(m)i)$, for $\bar{\alpha}(m)$ and for $\alpha = 0$.

imposing periodic boundary conditions over 3 lattice spacings, and use the resulting couplings then in any volume [4]. In contrast to a truncation in c-space, $\rho(r)$ vanishes smoothly at the edges of a unit hypercube, the normalization is automatically correct and the “decimation in p-space” is handy in perfect lattice perturbation theory.

The resulting couplings represent “smoothly smeared” lattice derivatives. Alternatively there are various ways to apply Symanzik’s program to cancel the standard $O(a^2)$ artifacts. Symanzik himself suggested to use additional couplings at distance 2 on the axes. These couplings are completely different from the perfect truncated ones. If, however, one constructs a Symanzik improved hypercube scalar, then the couplings look very similar [5]. The same behavior has been observed for staggered fermions [6]. Other Symanzik improved fermions in the literature (D234, Naik) are constructed along the axes again. However, I would rather recommend to use couplings in the unit hypercube, which is e.g. more promising for the restoration of rotational invariance.

In Fig. 3 we compare the *dispersion relations* of our hypercube scalar, the standard formulation and the Symanzik scalar on the axes. As in the fermionic case [4], the latter is good at small momenta, until it is hit by a “ghost” and then the solutions become complex, i.e. useless. The hypercube scalar, on the other hand, behaves very well all the way up to the edge of the Brillouin zone. This behavior persists at $m > 0$.

The hypercube scalar has also good *thermo-*

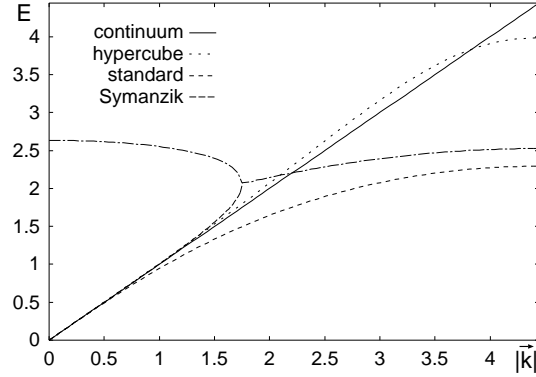


Figure 3. The spectrum along the (110) direction at $m = 0$.

dynamic properties. Fig. 4 shows the ratio pressure/(temperature)⁴, which is $\pi^2/90$ in the continuum. For a small number N_t of discrete points in Euclidean time, this ratio is approximated well for the hypercube scalar, whereas the standard action requires a large N_t to converge. Other quantities like the energy density look similar [5].

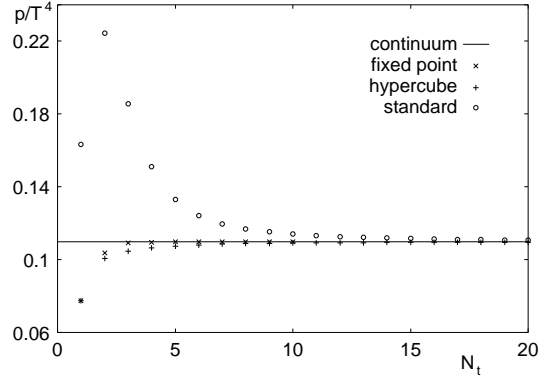


Figure 4. The ratio p/T^4 depending on N_t .

In thermodynamics it is fashionable to use *anisotropic* lattices. There is no additional problem to put a perfect action on an anisotropic lattice. In the most general case, lattice spacings (a_1, \dots, a_d) , we just have to substitute in eq. (2) $l \rightarrow (l_1/a_1, \dots, l_d/a_d)$, $\hat{k}_\mu \rightarrow (2/a_\mu) \sin(k_\mu a_\mu/2)$, and with $\bar{\alpha} \rightarrow \bar{\alpha}/a_\nu^2$ the mapping on the ν axis is ultralocal. In the typical case one has $a_d < a_s \equiv a_{\text{spatial}}$, and one might be afraid of time-like “ghosts”. One can avoid them by giving up

perfection in the temporal direction,

$$G_{ani}(k) = \sum_{\vec{l} \in \mathbb{Z}^{d-1}} \frac{\prod_{\nu=1}^{d-1} \hat{k}_\nu^2 / (k_\nu + 2\pi l_\nu / a_s)^2}{(\vec{k} + 2\pi \vec{l} / a_s)^2 + \hat{k}_d^2 + m^2} + \frac{\alpha}{a_s^2}.$$

We can also put the perfect action on a *triangular* lattice. In $d = 2$ we obtain a modified Π function, $\Pi_{tria}(k) =$

$$\frac{8[k_1 \cos \frac{k_1}{2} + k_2 \cos \frac{k_2}{2} - (k_1 + k_2) \cos \frac{k_1 + k_2}{2}]}{3k_1 k_2 (k_1 + k_2)},$$

where we refer to axes crossing under $\pi/3$. The blocking from the continuum then leads to

$$\phi(k) \sim \frac{3}{4} \sum_{l \in \mathbb{Z}^2} \varphi(k_l) \Pi_{tria}(k_l)$$

$$k_l \doteq k + \frac{4\pi}{3} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} l$$

$$G_{tria}(k) = \sum_{l \in \mathbb{Z}^2} \frac{\Pi_{tria}^2(k_l)}{k_{l,1}^2 + k_{l,2}^2 + k_{l,1} k_{l,2} + m^2} + \alpha.$$

Again the spectrum is perfect; we have e.g. full rotational invariance in the observables, although the lattice structure is visible in the action.

Back to the hypercubic lattice with spacing 1: we can also vary the convolution function in the blocking from the continuum,

$$\phi_x = \int dy \left[\prod_{\mu} f_n(x_\mu - y_\mu) \right] \varphi(y).$$

Sensible candidates are the B-spline functions $f_0(s) = \delta(s)$, $f_{n+1}(s) = \int_{s-1/2}^{s+1/2} f_n(t) dt$. The appropriate convolution function for the gauge field A_μ then reads $f_{n+1}(u_\mu) \prod_{\nu \neq \mu} f_n(u_\nu)$. This set of functions has the correct normalization and a “democracy of the continuum points”, which all contribute with the same weight to the lattice variables. In the propagator of eq. (2), the power of the Π function generalizes to $2n$. Decimation ($n = 0$) fails because the sum over l diverges in $d > 1$, and $n = 1$ is what we had before. For higher n we can still achieve ultralocality in $d = 1$ by adding kinetic terms in α , but the locality in $d > 1$ is best for $n = 1$. Generally, overlapping blocks seem to be unfavorable for locality. We can contract f_n into $[-1/2, 1/2]$, and re-adjust the normalization (giving up “democracy”). Thus f_2 turns into an “Eiffel tower function” $\tilde{f}_2(s) = (2 - 4|s|)\Theta(1/2 - |s|)$, which is in business in view of the induced locality [5]. For

$n > 2$ one obtains blocking schemes which are difficult to relate to finite blocking factors, hence they are not easily compatible with the multigrid improvement. So I recommend the “Eiffel tower function” \tilde{f}_2 as a promising alternative to f_1 .

Finally we proceed to the $\lambda\phi^4$ theory. If we block perturbatively from the continuum, we encounter divergent loop integrals, which can be regularized in the continuum. There is no divergence, however, for the *anharmonic oscillator* ($d = 1$). There we constructed an $O(\lambda)$ perfect action. It involves 2- and 4-variable couplings $\propto \lambda$, which have an analytic form in momentum space and which we evaluated numerically in c-space. For $\bar{\alpha}(m)$ they do not couple any variables over distance > 2 (the perfect action to $O(\lambda^n)$ extends to maximal distances $2n$ ($n \geq 1$)).

The performance of this action is of interest in view of the direct application of the perfect quark-gluon vertex function in QCD [7]. We measured the first two energy gaps [8], ΔE_1 and ΔE_2 , for our action and for the standard action, and observed that they are clearly closer to the continuum results for the new action up to $\tilde{\lambda} \doteq \lambda/m^3 \simeq 0.25$. This corresponds to an improved asymptotic scaling. However, for the scaling quantity $\Delta E_2/\Delta E_1$ the improvement is difficult to demonstrate, because it is restricted to tiny $\tilde{\lambda}$. In both cases, the $O(\tilde{\lambda})$ perfect action is successful up to the magnitude of $\tilde{\lambda}$, where also first order continuum perturbation theory – for the considered observable – collapses.

REFERENCES

1. W. Bietenholz and U.-J. Wiese, Nucl. Phys. B464 (1996) 319.
2. A. Tsapalis and U.-J. Wiese, Nucl. Phys. B (Proc. Suppl.) 53 (1997) 948.
3. T. Bell and K. Wilson, Phys. Rev. D11 (1975) 3431.
4. W. Bietenholz, R. Brower, S. Chandrasekharan and U.-J. Wiese, Nucl. Phys. B (Proc. Suppl.) 53 (1997) 921.
5. W. Bietenholz, in prep.
6. F. Karsch, hep-lat/9706006.
7. K. Orginos et al., hep-lat/9709100.
8. W. Bietenholz and T. Struckmann, in prep.